Ex :- Find a P.I of the equation

$$(D^2 - D')z = A\cos(lx + my)$$

where A, l, m are constants.

Sol^n :-

$$P.I = \frac{1}{D^2 - D'} A\cos(lx + my)$$

$$= A \cdot \frac{1}{D^2 - D'} \cos(lx + my)$$

$$= A \cdot \frac{1}{-l^2 - D'} \cos(lx + my)$$

$$= -A\left(\frac{1}{D' + l^2}\right) \cos(lx + my)$$

$D^2 = -l^2$

$D'^2 = -m^2$

$$= -A\left(\frac{D' - l^2}{D'^2 - l^4}\right) \cos(lx + my)$$

$$= -A\left(\frac{D' - l^2}{-m^2 - l^4}\right) \cos(lx + my)$$

$$= \frac{A}{m^2 + l^4}\left[-m\sin(lx + my) - l^2\cos(lx + my)\right]$$

$$= \frac{-A}{m^2 + l^4}\left[m\sin(lx + my) + l^2\cos(lx + my)\right]$$

5. Equations with variable co-efficients

consider equations of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \longrightarrow (1)$$

which ~~may~~ may be written in the form

$$L(z) + f(x, y, z, p, q) = 0 \longrightarrow (2)$$

where 'L' is the differential operator defined by the equation

$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} \longrightarrow (3)$$

in which R, S, T are continuous functions of x and y possessing continuous partial derivatives of ~~...~~ as high an order as necessary. By a suitable change of the independent variables we shall show that any equation of the type (2) can be reduced to one of three canonical forms.

Suppose we change the independent variables from x, y to ξ, η where

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

and we write $z(x, y)$ as $\zeta(\xi, \eta)$, then equation (1) takes the form —

$$A(\xi_x, \xi_y)\frac{\partial^2 \zeta}{\partial \xi^2} + 2B(\xi_x, \xi_y; \eta_x, \eta_y)\frac{\partial^2 \zeta}{\partial \xi \partial \eta}$$

$$+ A(\eta_x, \eta_y)\frac{\partial^2 \zeta}{\partial \eta^2} = F(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \longrightarrow (4)$$

where $A(u, v) = Ru^2 + Suv + Tv^2 \longrightarrow (5)$

and $B(u_1, v_1; u_2, v_2) = R u_1 u_2 + \frac{1}{2} S (u_1 v_2 + u_2 v_1) + T v_1 v_2 \longrightarrow (6)$.

and the function F is readily derived from the given function f.

The problem now is to determine ξ and η so that equation (4) takes the simplest possible form.

Case (a) : $S^2 - 4RT > 0$

when this condition is satisfied, the roots λ_1, λ_2 of the equation $R\alpha^2 + S\alpha + T = 0 \rightarrow (7)$ are real and distinct, and the co-efficients of $\frac{\partial^2 u}{\partial \xi^2}$ and $\frac{\partial^2 u}{\partial \eta^2}$ in equation (4) will vanish if we chose ξ and η s.t.

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}, \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}$$

A suitable choice would be

$$\xi = f_1(x, y), \quad \eta = f_2(x, y) \longrightarrow (8)$$

where $f_1 = c_1$, $f_2 = c_2$ are the solutions of the first-order O.D.E

$$\frac{dy}{dx} + \lambda_1(x, y) = 0, \quad \frac{dy}{dx} + \lambda_2(x, y) = 0 \rightarrow (9)$$

respectively.

In general,

$$A(\xi_x, \xi_y)\, A(\eta_x, \eta_y) - B^2(\xi_x, \xi_y ; \eta_x, \eta_y) = (4RT - S^2)(\xi_x \eta_y - \xi_y \eta_x)^2 \quad \rightarrow (10).$$

so that when the A's are zero ~~or~~

$$B^2 = (S^2 - 4RT)(\xi_x \eta_y - \xi_y \eta_x)^2$$

and since $S^2 - 4RT > 0$, it follows that $B^2 > 0$ and therefore that ~~we~~ we may divide both sides of the equation by it. Hence if we make the substitutions defined by the equations (8) and (9), we find that equation (1) is reduced to the form

$$\frac{\partial^2 \zeta}{\partial \xi\, \partial \eta} = \phi(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \quad \rightarrow (11)$$

Ex :- Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$ to canonical form.

Sol^n :- Here $R = 1$, $S = 0$, $T = -x^2$.

$$S^2 - 4RT = 4x^2$$

$$\frac{dy}{dx} = \frac{S \pm \sqrt{S^2 - 4RT}}{2R} = \pm \frac{2x}{2} = \pm x.$$

$$\xi = y + \frac{x^2}{2}, \quad \eta = y - \frac{x^2}{2}.$$

$$A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$$

$$= Rx^2 + S \cdot x \cdot + T$$

$$= x^2 - x^2 = 0$$

$$A(\eta_x, \eta_y) = R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2$$

$$= Rx^2 - Sx + T$$

$$= x^2 - x^2 = 0.$$

$$B(\xi_x, \xi_y ; \eta_x, \eta_y)$$

$$= R\xi_x\eta_x + \frac{S}{2}\left(\xi_x\eta_y + \xi_y\eta_x\right) + T\xi_y\eta_y$$

$$= -x^2 - x^2 = -2x^2$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$\xi = y + \frac{x^2}{2}$

$\eta = y - \frac{x^2}{2}$

$$= \frac{\partial z}{\partial \xi} \cdot x + \frac{\partial z}{\partial \eta} \cdot (-x)$$

$\xi_x = x$

$\eta_x = -x$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$z_{xx} = z_{\xi\xi}\,\xi_x^2 + 2 z_{\xi\eta}\,\xi_x \eta_x + z_{\eta\eta}\,\eta_x^2 + z_\xi\,\xi_{xx} + z_\eta\,\eta_{xx}$$

$$= z_{\xi\xi} x^2 + 2 z_{\xi\eta}(-x^2) + z_{\eta\eta} x^2 + z_\xi - z_\eta$$

$$z_{yy} = z_{\xi\xi}\,\xi_y^2 + 2 z_{\xi\eta}\,\xi_y \eta_y + z_{\eta\eta}\,\eta_y^2 + z_\xi\,\xi_{yy} + z_\eta\,\eta_{yy}$$

$$= z_{\xi\xi} + 2 z_{\xi\eta} + z_{\eta\eta}$$

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow \cancel{z_{\xi\xi} x^2} - 2x^2 z_{\xi\eta} + \cancel{z_{\eta\eta} x^2} + z_\xi - z_\eta$$

$$= \cancel{x^2 z_{\xi\xi}} + 2x^2 z_{\xi\eta} + \cancel{x^2 z_{\eta\eta}}$$

~~$x^2 = \xi - \eta$~~

$$\Rightarrow 4x^2 z_{\xi\eta} = z_\xi - z_\eta$$

$$\Rightarrow \frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{1}{4(\xi - \eta)}(z_\xi - z_\eta)$$

Finally, we replace z by ξ.

Case (b) : $S^2 - 4RT = 0$. In such circumstances, the roots of equation (7) are equal. We define the function ξ precisely as in case (a) and take 'η' to be any function of x, y which is independent of ξ.

We then have, $A(\xi_x, \xi_y) = 0$ and hence from equation (10), $B(\xi_x, \xi_y ; \eta_x, \eta_y) = 0$.

on the other hand, $A(\eta_x, \eta_y) \neq 0$; otherwise

η would be a function of ξ.

Putting $A(\xi_x, \xi_y)$ and B equal to zero and dividing by $A(\eta_x, \eta_y)$, the canonical form of equation (1) is, in this case,

$$\frac{\partial^2 \psi}{\partial \eta^2} = \varphi(\xi, \eta, \psi, \psi_\xi, \psi_\eta) \longrightarrow (12).$$

Ex :- Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form and hence solve it.

Soln :- $R = 1$, $S = 2$, $T = 1$.

$$\frac{B \pm \sqrt{B^2 -}}{2A}$$

$$\frac{S \pm \sqrt{S^2 - 4}}{2R}$$

$$\frac{2}{2}$$

$$\alpha^2 + 2\alpha + 1 = 0.$$

$$\frac{dy}{dx} = \frac{S \pm \sqrt{S^2 - 4RT}}{2R} = \frac{2}{2} = 1$$

$$\Rightarrow dx = dy \Rightarrow x - y = c_1$$

$\xi = x - y, \quad \eta = x + y.$

$$A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$$

$$= 1 + 2(-1) + 1.(+1) = 0$$

$$A(\eta_x, \eta_y) = R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2.$$

$$= 1 + 2 + 1 = 4$$

$$B(\xi_x, \xi_y; \eta_x, \eta_y) = R\xi_x\eta_x + \frac{S}{2}(\xi_x\eta_y + \xi_y\eta_x) + T\xi_y\eta_y$$

$$= 1 + 1(1-1) + 1.(-1) = 0.$$

$$\therefore \frac{\partial^2 \psi}{\partial \eta^2} = 0.$$

Thus, the solution of the above equation is $\psi = \eta f_1(\xi) + f_2(\xi)$. where the functions f_1 and f_2 are arbitrary. Hence the original equation has solution $z = (x+y) f_1(x-y) + f_2(x-y)$.

case (c) : $S^2 - 4RT < 0$. This is formally the same as case (a) except that now the roots of equation (7) are complex. If we go through the procedure outlined in case (a), we find that the equation (1) reduces to the form (11) but that the variables ξ, η are not real but are in fact complex conjugates.

To get a real canonical form, we make the further transformation

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{i}{2}(\eta - \xi)$$

Thus, $$\frac{\partial^2 u}{\partial \xi \, \partial \eta} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2}\right)$$

The desired canonical form is —

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = \varphi(\alpha, \beta, u, u_\alpha, u_\beta)$$

Ex :- Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

$\frac{B \pm \sqrt{B^2 - 4AC}}{2A}$

to canonical form

Soln :- $R = 1, \quad S = 0, \quad T = x^2$

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4x^2}}{2} = \pm\, i x.$$

$$\xi = iy + \frac{x^2}{2}, \quad \eta = -iy + \frac{x^2}{2}$$

$$\alpha = \frac{1}{2}(\xi + \eta) = \frac{x^2}{2}.$$

$$\beta = \frac{i}{2}(\eta - \xi) = \frac{i}{2}(-2iy) = y.$$

$-iy + \frac{x^2}{2} = c$

$$A(\alpha_x, \alpha_y) = R\alpha_x^2 + S\alpha_x\alpha_y + T\alpha_y^2$$
$$= x^2$$

$$A(\beta_x, \beta_y) = R\beta_x^2 + S\beta_x\beta_y + T\beta_y^2$$
$$= x^2$$

$$B(\alpha_x, \alpha_y; \beta_x, \beta_y) = R\alpha_x\beta_x + \frac{1}{2}S(\alpha_x\beta_y + \alpha_y\beta_x) + T\alpha_y\beta_y = 0.$$

$$\cancel{\alpha = \frac{x^2}{2}}$$
$$\cancel{\alpha_x = x}$$
$$\cancel{\beta = y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x}.$$
$$= \frac{\partial z}{\partial \alpha}\cdot x.$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \beta}$$

$$z_{xx} = z_{\alpha\alpha}\alpha_x^2 + 2z_{\alpha\beta}\alpha_x\beta_x + z_{\beta\beta}\beta_x^2 + z_\alpha\alpha_{xx} + z_\beta\beta_{xx}.$$
$$= z_{\alpha\alpha}\cdot x^2 + z_\alpha\cdot 1.$$

$$z_{yy} = z_{\alpha\alpha}\alpha_y^2 + 2z_{\alpha\beta}\alpha_y\beta_y + z_{\beta\beta}\beta_y^2 + z_\alpha\alpha_{yy} + z_\beta\beta_{yy}$$
$$= z_{\beta\beta}$$

Thus, the given equation becomes —

$$x^2 z_{\alpha\alpha} + z_\alpha + x^2 z_{\beta\beta} = 0$$

$$\Rightarrow x^2 \left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} \right) = - \; \frac{\partial z}{\partial \alpha}$$

$$\Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{-1}{2\alpha} \frac{\partial z}{\partial \alpha}$$

Finally, we replace z by y.

$$\frac{\partial^2 y}{\partial \alpha^2} + \frac{\partial^2 y}{\partial \beta^2} = -\frac{1}{2\alpha} \frac{\partial y}{\partial \alpha}$$