§ Let (X,d) be a metric space and Y a subspace of X. Let $z \in Y$ and $r>0$.

Then $$S_Y(z,r) = S_X(z,r) \cap Y$$

where $S_Y(z,r)$ and $S_X(z,r)$ denotes the ball with centre z and radius r in Y and X respectively.

Proof:- $$S_X(z,r) \cap Y = \{x \in X : d(x,z) < r\} \cap Y$$
$$= \{x \in Y : d(x,z) < r\}$$
$$= S_Y(z,r), \text{ since } Y \subseteq X$$ //

Theorem: Let (X,d) be a metric space and Y a subspace of X. Let $Z \subseteq Y$. Then

(i) Z is open in Y if and only if $\exists$ an open set $G \subseteq X$ s.t $Z = G \cap Y$.

(2) Z is closed in Y if and only if $\exists$ a closed set $F \subseteq X$ s.t. $Z = F \cap Y$.

<u>Proof</u>: (1) Let Z be open in Y.

Then, if z is any point of Z, $\exists$ an open ball $S_Y(z, r) \subseteq Z$. We observe that the radius r of the ball $S_Y(z, r)$ depends on the point $z \in Z$. We then have,

$$Z = \bigcup_{z \in Z} S_Y(z, r)$$

$$= \bigcup_{z \in Z} S_X(z, r) \cap Y \quad \text{(using previous result)}$$

$$= \left(\bigcup_{z \in Z} S_X(z, r) \right) \cap Y$$

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$$= G \cap Y, \text{ where } G = \bigcup_{z \in Z} S_X(z, r)$$

is open in X.

On the other hand, suppose $Z = G \cap Y$ where G is open in X. If $z \in Z$, then z is a point of G and so there exists

an open ball $S_X(z, r)$ such that $S_X(z, r) \subseteq G$.
Hence,
$$S_Y(z, r) = S_X(z, r) \cap Y \quad \text{(By previous result)}$$
$$\subseteq G \cap Y = Z,$$
so that z is an interior point of the subset Z of Y.
As z is an arbitrary point of Z, it follows that Z is open in Y.

(ii) Z is closed in Y if and only if $(X/Z) \cap Y$ is open in Y.
Hence, Z is closed in Y if and only if there exists an open set G in X such that
$$(X \setminus Z) \cap Y = G \cap Y \quad \text{using (1) above}$$

On taking complements in X on both sides, we have
$$Z \cup (X \setminus Y) = (X \setminus G) \cup (X \setminus Y)$$

Hence,
$$Z = Z \cap Y = (Z \cup (X \setminus Y)) \cap Y$$
$$= ((X \setminus G) \cup (X \setminus Y)) \cap Y$$
$$= (X \setminus G) \cap Y$$
So, Z is the intersection of the closed set $X \setminus G$ and Y.
Conversely,
let $Z = F \cap Y$, where F is closed in X. Then
$X \setminus Z = (X \setminus F) \cup (X \setminus Y)$ and so
$$(X \setminus Z) \cap Y = ((X \setminus F) \cup (X \setminus Y)) \cap Y = (X \setminus F) \cap Y,$$
where $X \setminus F$ is open in X.
Hence, $(X \setminus Z) \cap Y$ is open in Y i.e, Z is closed in Y. //

§. Let Y be a subspace of a metric space (X, d).

(1) Every subset of Y that is open in Y is also open in X if Y is open in X.

(2) Every subset of Y that is closed in Y is also closed in X if and only if Y is closed in X.

Proof: (1) Suppose every subset of Y open in Y is also open in X. We want to show that Y is open in X. Since Y is an open subset of Y, it must be open in X.

Conversely, Suppose Y is open in X. Let Z be an open subset of Y. By previous result (1), $\exists$ an open subset G of X such that $Z = G \cap Y$.

Since G and Y are both open subsets of X, their intersection must open in X i.e., Z must be open in X.

(2) ~~P.~~ Every subset of Y that is closed in Y is also closed in X. To show Y is closed in X.

Let us consider Y itself as a subset of Y. In any metric space, the space itself is always closed.

$\therefore$ Y is closed in Y

$\therefore$ Y is closed in X,

Conversely,
Let Y is closed in X.
To show every subset $F \subseteq Y$ which is closed in Y is closed in X.
Let F be a subset of Y that is closed in Y.
We know F is closed in Y iff $\exists$ a set P that is closed in X s.t $F = Y \cap P$.
$\therefore$ P is also closed in X.
We know, that the intersection of two closed sets in a metric space is also closed.
Since both Y and P are closed in X, their intersection $Y \cap P$ must be closed in X.
Therefore, F is closed in X.