Q State the principle of conservation of linear momentum.

Solⁿ:- In absence of an external force, the total linear momentum of a system of particles remains constant. ie If $\vec{F} = 0$ then.

$$m\frac{d\vec{v}}{dt} = \vec{F} = 0 \Rightarrow m\vec{v} = \text{constant}.$$

Q Define amplitude and frequency of a ~~simp~~ SHM

Solⁿ:- Amplitude:- Maximam displacement of the particle from its mean position.

Frequency:- Number of complete oscillation per second.

$$f = \frac{1}{T}.$$

Q Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another.

Solⁿ:- Let three equal forces F act a point O. For equilibrium, using Lami's theorem

$$\frac{F}{\sin\alpha} = \frac{F}{\sin\beta} = \frac{F}{\sin\gamma} \Rightarrow \sin\alpha = \sin\beta = \sin\gamma$$

$$\Rightarrow \alpha = \beta = \gamma.$$

Q Prove that the change of K.E of a body is equal to work done.

Soln: From Newton's 2nd law.

$$m\frac{dv}{dt} = F.$$

$\therefore$ Workdone $= \int_0^T F \, d\vec{r}$

$= \int_0^T m\frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt} \, dt.$

$= \int_0^T m\frac{d\vec{v}}{dt} \cdot \vec{v} \, dt.$

$= \left[\frac{1}{2}mv^2\right]_0^T$

$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$ $\left[\begin{array}{l} v_1 \to \text{initial velocity} \\ v_2 \to \text{final velocity} \end{array}\right]$

$\therefore$ Workdone = change in K.E.

Q Two forces P and Q acting on a particle at an angle α, have a result $(2k+1)\sqrt{P^2+Q^2}$. When they act at an angle $90° - \alpha$. The result becomes $(2k-1)\sqrt{P^2+Q^2}$. Prove that $\tan\alpha = \frac{k-1}{k+1}$

Soln:- Resultant of two forces R is given by

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

when, $\theta = \alpha$. $(2k+1)^2(P^2+Q^2) = P^2 + Q^2 + 2PQ\cos\alpha$.

$\Rightarrow \{(2k+1)^2 - 1\}P^2 + Q^2 = 2PQ\cos\alpha.$

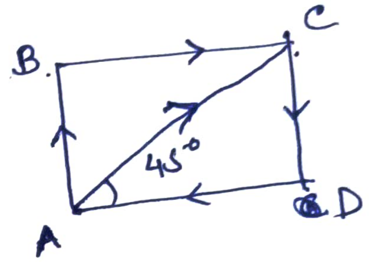
When $\theta = 90-\alpha$.

$$(2k-1)^2(P^2+Q^2) = P^2+Q^2+2PQ\cos(90-\alpha)$$

$$\{(2k-1)^2-1^2\}(P^2+Q^2) = 2PQ\sin\alpha.$$

$$\therefore \tan\alpha = \frac{(2k-1)^2-1^2}{(2k+1)^2-1^2} = \frac{4k^2-4k}{4k^2+4k} = \frac{k-1}{k+1}.$$

Q Forces of magnitude 1, 2, 3, 4, $2\sqrt{2}$ act along sides AB, BC, CD, DA and AC of the square ABCD. Show that their resultant is a couple, find the moment.



Solⁿ:- Taking A as origin, AD as x-axis, AB as Y axis,

Let length of the sides in a.

Resolved parts along x axis

$$X = 2-4+2\sqrt{2}.\cos 45^\circ = 0.$$

Resolved parts along Y axis.

$$Y = 1-3+2\sqrt{2}\sin 45^\circ = 0. \quad \therefore R=\sqrt{X^2+Y^2}=0$$

Sum of Moments about A,

$$G = 1\times 0 + 2\times a + 3\times a + 4\times 0 = 5a$$

$\therefore$ Since $R=0$, $G\neq 0$, it reduces to a couple with moment a.

Q A particle moves towards a centre of attraction starting from rest at a distance a from centre. If its velocity at any distance x from the centre vary as $\sqrt{\frac{a^2-x^2}{x^2}}$, find the law of force.

Solⁿ:- Given $v = k\sqrt{\frac{a^2-x^2}{x^2}}$ ($k \to$ constant)

$$\Rightarrow v^2 = k^2\left(\frac{a^2-x^2}{x^2}\right) \to (1)$$

Now $f = m\frac{du}{dt} = m\frac{du}{dx}\frac{dx}{dt}$

$= m\frac{du}{dx}u.$ $\quad [\frac{dx}{dt} = u]$.

$= mu\frac{du}{dx}.$

differentiating (1) w.r.to x $\quad 2u\frac{du}{dx} = k^2\left[\frac{-2a^2}{x^3}\right]$

So, $\quad f = m\frac{k^2}{2}\left(\frac{-2a^2}{x^3}\right) = -\frac{mk^2a^2}{x^3}$

$\therefore$ ~~For~~ $\quad f \propto \frac{1}{x^3}$ #.

Q A particle of mass m is projected vertically upward under gravity, the resistance of the air being mk times of velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}[\lambda - \log(1+\lambda)]$, where $V \to$ terminal velocity, $\lambda V \to$ intial velocity

Solⁿ:- Eqⁿ of motion ~~is~~ for upward motion is

$m\frac{d^2x}{dt} = -mg - mku.$

$\Rightarrow \frac{d^2x}{dt} = -g - ku.$ $\quad \left[u = \frac{dx}{dt}\right].$

$\Rightarrow \frac{du}{dt} = -g - ku$

By defⁿ terminal velocity is that when accⁿ = 0 in downward direction ie $\frac{du}{dt} = 0$ when $u = -V$

$\therefore -g + kV = 0 \Rightarrow k = \frac{g}{V}.$

Hence equation of motion becomes

$\frac{du}{dt} = -g - \frac{g}{V}u.$

$$\Rightarrow \quad \frac{dv}{dt} = -g\left(\frac{V+v}{V}\right)$$

$$\Rightarrow \quad \cancel{\frac{dv}{V+v}} \quad \frac{dv}{dx}\cdot\frac{dx}{dt} = -g\,\frac{V+v}{V}$$

$$\Rightarrow \quad v\frac{dv}{dx} = -g\left(\frac{V+v}{V}\right)$$

$$\Rightarrow \quad \frac{v\,dv}{V+v} = -\frac{g}{V}\,dx.$$

Integrating, $v - V\log(V+v) = -\frac{g}{V}x + C_1$

when $t=0$, $x=0$, $v=\lambda V$

So $\quad \lambda V - V\log(V+\lambda V) = -\frac{g}{V}\times 0 + C_1$

$$\therefore \quad C_1 = \lambda V - V\log(V+\lambda V).$$

Let H be the greatest height, then $v=0$, $x=$

$$0 - V\log V = -\frac{g}{V}H + C_1$$

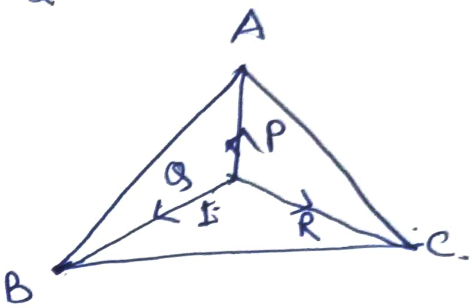
$$\Rightarrow \quad -V\log V = -\frac{g}{V}H + \lambda V - V\log(V+\lambda V)$$

$$\Rightarrow \quad \frac{g}{V}H = \lambda V - V\log\left(\frac{V+\lambda V}{V}\right)$$

$$\rightarrow \quad H = \frac{V^2}{g}\left[\lambda - \log(1+\lambda)\right]$$

Q Forces P, Q, R acting along IA, IB, IC where I is the incentre of a triangle ABC are in equilibrium, Prove that

$$P : Q : R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$$



Solⁿ:- At I.

$$\angle BIC = 180^\circ - \frac{1}{2}\angle B - \frac{1}{2}\angle C$$

$$= 180^\circ - \frac{1}{2}(\angle B + \angle C)$$

$$= 180^\circ - \frac{1}{2}(180^\circ - \angle A) = 90^\circ + \frac{1}{2}\angle A.$$

Similarly $\angle CIA = 90^\circ + \frac{1}{2}\angle B$

$\angle AIB = 90^\circ + \frac{1}{2}\angle C.$

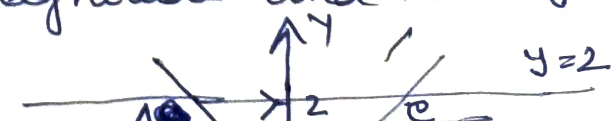
By Lami's theorem

$$\frac{P}{\sin\angle BIC} = \frac{Q}{\sin\angle CIA} = \frac{R}{\sin\angle AIB}$$

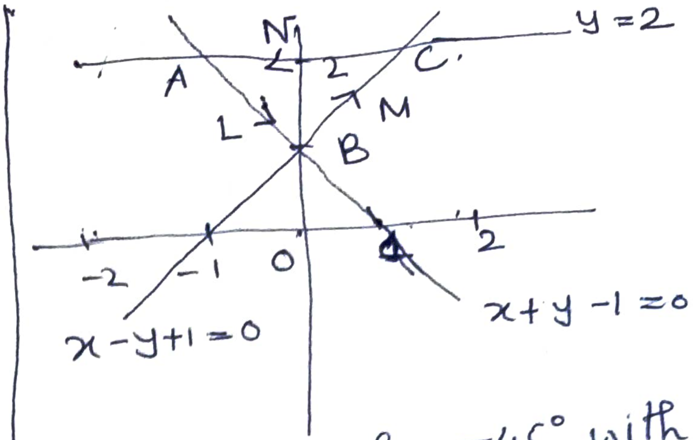
$$\Rightarrow \frac{P}{\sin(90 + A/2)} = \frac{Q}{\sin(90 + \frac{B}{2})} = \frac{R}{\sin(90 + C/2)}$$

$$\Rightarrow \frac{P}{\cos A/2} = \frac{Q}{\cos B/2} = \frac{R}{\cos C/2}.$$

Q Forces L, M, N act along the sides of a triangle formed by the lines $x+y-1=0$, $x-y+1=0$ and $y=2$. Find the ~~Magdi~~ magnitude and line of action of the resultant.



Solⁿ :- Let the forces L M and N act along AB, BC and CA as shown in figure.



The slope of the line $x+y-1=0$ is (-1). It makes an angle $-45°$ with x-axis. Similarly the line $x-y+1=0$ makes an angle $45°$ with x axis and the line $y=2$ makes $0°$ with x axis.

Resolving about x and y axis.

$$X = L\cos(-45°) + M\cos 45° + N\cos 0°$$
$$= \frac{L}{\sqrt{2}} + \frac{M}{\sqrt{2}} + N$$
$$Y = L\sin(-45°) + M\sin(45°) + N\sin 0$$
$$= -\frac{L}{\sqrt{2}} + \frac{M}{\sqrt{2}}.$$

Resultant $R = \sqrt{X^2+Y^2} = \sqrt{\left(\frac{L+M}{\sqrt{2}}+N\right)^2 + \left(\frac{M-L}{\sqrt{2}}\right)^2}$. #

Let the resultant passes through (x,y). Then moment of the resultant about (x,y) is zero.

Moment of resultant about (x,y) = sum of moment of the forces (x,y) $= 0$.

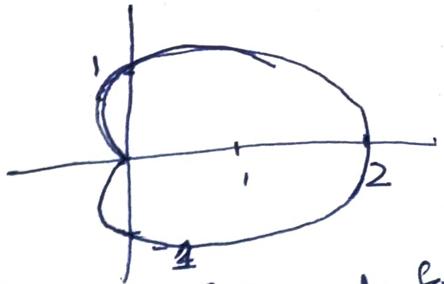
i.e. $$\frac{x+y-1}{\sqrt{1^2+1^2}} \cdot L + \frac{x-y+1}{\sqrt{1^2+1^2}} M + (y-2) N = 0$$

$$\Rightarrow \left(\frac{L}{\sqrt{2}} + \frac{M}{\sqrt{2}}\right)x + \left(\frac{L}{\sqrt{2}} - \frac{M}{\sqrt{2}} + N\right)y - \left(\frac{L}{\sqrt{2}} + \frac{M}{\sqrt{2}} - 2N\right) = 0$$

is the eqⁿ of line of action.

Q Find the C.G of the area of the cardiod.

$r = a(1+\cos\theta)$.



Sol^n:-

$$\cancel{\bar{x} = \frac{\int_{\theta=0}^{2\pi}\int r\cos\theta\, r\,dr\,d\theta}{\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r\,dr\,d\theta}}$$

$$\bar{x} = \frac{\int_{\theta=0}^{2\pi}\int_{0}^{r} r\cos\theta\, r\,dr\,d\theta}{\int_{\theta=0}^{2\pi}\int_{0}^{r} r\,dr\,d\theta}$$

[formula for polar co-ordinate]

$$= \frac{\int_0^{2\pi} a^3\cos\theta \cdot \frac{r^3}{3}\,d\theta}{\int_0^{2\pi} \frac{a^2 r^2}{2}\,d\theta}$$

$$= \frac{a\frac{1}{3}\int_0^{2\pi}\cos\theta\cdot(1+\cos\theta)^3\,d\theta}{\frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\,d\theta}$$

[Integration do yourself].

$$= \frac{\frac{1}{3}a\left[\frac{15\pi}{4}\right]}{\frac{1}{2}\cdot 3\pi} = \frac{\cancel{2}a\,\cancel{15}^{5}\cancel{\pi}}{\cancel{3}\times 3\cancel{\pi}\times \cancel{4}_2}$$

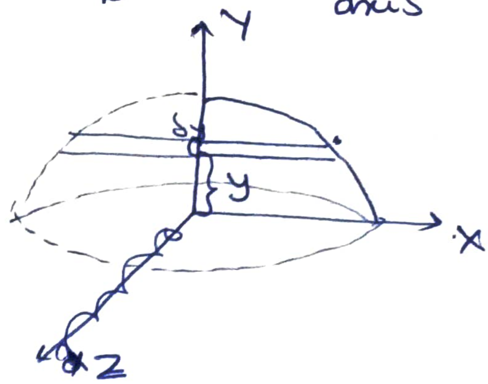
$$= \frac{5a}{6}.$$

$\bar{y} = 0$. due to symmetry.

$\therefore$ Co-ordinate of C.G is $\left(\frac{5a}{6}, 0\right)$.

Q Find C.G of the solid formed by revolution of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, about minor axis

Sol^n:- Let us cut a disk of breadth δy at a distance y from the centre.



~~Area of~~

Volume of a disc.

$$= \rho\pi x^2 \delta y \qquad [\rho \to \text{density}].$$

$$\therefore \bar{y} = \frac{\int_0^b y\rho\pi x^2 dy}{\int_0^b \rho\pi x^2 dy}$$

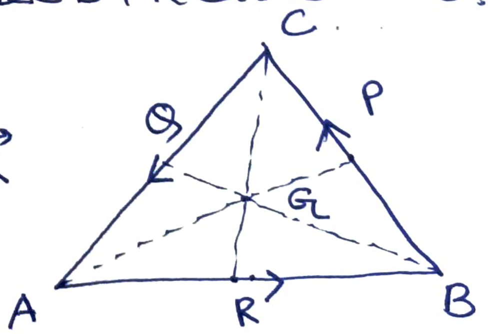
$$= \frac{\int_0^b a^2 y \left(1-\frac{y^2}{b^2}\right) dy}{\int_0^b a^2 \left(1-\frac{y^2}{b^2}\right) dy} \qquad \left[\text{Integrate yourself}\right].$$

$$= \frac{3b}{8}.$$

Due to symmetry. $\bar{x}=0$, $\bar{z}=0$.

Co-ordinate of C.G of the hemisphere is $\left(0, \frac{3b}{8}, 0\right)$.

Q Three forces. P, Q, R act in the same sense along the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of the triangle ABC. Show that, if their resultant passes through the centroid, then $P\operatorname{cosec}A + Q\operatorname{cosec}B + R\operatorname{cosec}C = 0$.



Sol^n:- Let G be the centroid.

Let $\vec{F}$ the resultant of $\vec{P}, \vec{Q}, \vec{R}$

$\vec{F}$ passes through G.

Then the moment of all forces about G must be zero.

Q The velocities of a particle ~~along~~ along and perpendicular to the radius from a fixed point are λr and $\mu\theta$. Find the path. Also find the radial and cross radial accn.

Soln:- Given radial velocity $\frac{dr}{dt} = \lambda r$.

and cross radial velocity $r\frac{d\theta}{dt} = \mu\theta$.

$$\text{So, } \frac{dr}{d\theta} = \frac{\frac{dr}{dt}}{\frac{d\theta}{dt}} = \frac{\lambda r}{\mu\theta/r} = \frac{\lambda r^2}{\mu\theta}.$$

$$\Rightarrow \frac{dr}{r^2} = \frac{\lambda}{\mu}\left(\frac{d\theta}{\theta}\right).$$

Integrating. $-\frac{1}{r} = \frac{\lambda}{\mu}\log\theta + C_1$

$$\Rightarrow r = \frac{1}{C_1 - \lambda/\mu \log\theta}.$$

Radial accn, $\frac{d^2r}{d\theta^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$= \frac{d}{dt}\left(\frac{dr}{d\theta}\right) - r\left(\frac{\mu\theta}{r}\right)^2$$

$$= \frac{d}{dt}(\lambda r) - r\frac{\mu^2\theta^2}{r^2}$$

$$= \lambda\frac{dr}{dt} - \frac{\mu^2\theta^2}{r}$$

$$= \lambda^2 r - \frac{\mu^2\theta^2}{r}.$$

Cross radial accn.

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = r\frac{d}{dt}\left(\frac{d\theta}{dt}\right) - 2\frac{dr}{dt}\frac{d\theta}{dt}.$$

$$= r\frac{d}{dt}\left(\frac{\mu\theta}{r}\right) - 2\lambda r\frac{\mu\theta}{r}.$$

$$= r\frac{\left\{r\mu\frac{d\theta}{dt} - \mu\theta\frac{dr}{dt}\right\}}{r^2} - 2\lambda r\frac{\mu\theta}{r}$$

$$= \mu\frac{d\theta}{dt} - \frac{\mu\theta}{r}\frac{dr}{dt} - 2\lambda\mu\theta$$

$$= \mu\frac{\mu\theta}{r} - \frac{\mu\theta}{r}\lambda r + 2\lambda\mu\theta.$$

$$= \frac{\mu^2\theta}{r} - \lambda\mu\theta + 2\lambda\mu\theta. \#$$

Q Prove that C.G of a body is unique.

Proof:- Suppose a body has two different C.G's G_1 and G_2. The Weight W of the body acts at G_1. ~~So, resultant mom moment of W about any~~. Also W acts through G_2. Hence $\overrightarrow{G_2G_1}$ is parallel to W.

But we can choose W in any direction by rotating the body. So $\overrightarrow{G_2G_1}$ is parallel to W if $\overrightarrow{G_2G_1}$ is $\vec{0}$. Hence G_2 as G_1 is same point.

$\therefore$ C.G of the body is unique.

Q. An impulse I changes the velocity of a particle of mass m from v_1 ~~at~~ to v_2. Show that

$$KE = \frac{1}{2} I (v_1 + v_2).$$

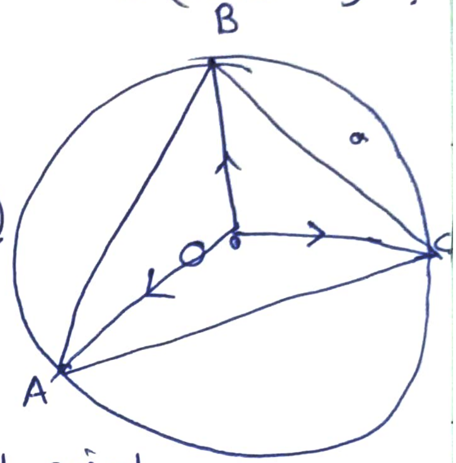
Soln:- Impulse $I = m(v_2 - v_1)$. $\longrightarrow$ (1).

$$KE = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$= \frac{1}{2} m (v_2 - v_1)(v_2 + v_1) = \frac{1}{2} I (v_2 + v_1) \quad \#.$$

Q Force $\vec{P}, \vec{Q}, \vec{R}$ acting along $\vec{OA}, \vec{OB}, \vec{OC}$ where O is the circumcentre of the triangle ABC. is equilibrium. Show that

$$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}.$$



Solⁿ:- At circumcentre O.

$OA = OB = OC = r$ (circumradius)

$\therefore \angle BOC = 2A.$

$\angle COA = 2B.$

$\angle AOB = 2C.$ by property of circle.

By Lami's theorem.

$$\frac{P}{\sin\angle BOC} = \frac{Q}{\sin\angle COA} = \frac{R}{\sin\angle AOB}$$

$$\Rightarrow \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}.$$

So $\sin A = \frac{a}{2R}$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}.$

$$\therefore \sin 2A = 2\,\frac{a}{2R}\,\frac{b^2+c^2-a^2}{2bc}.$$

Similarly $\sin 2B = \frac{b(c^2+a^2-b^2)}{2R\,ca}.$

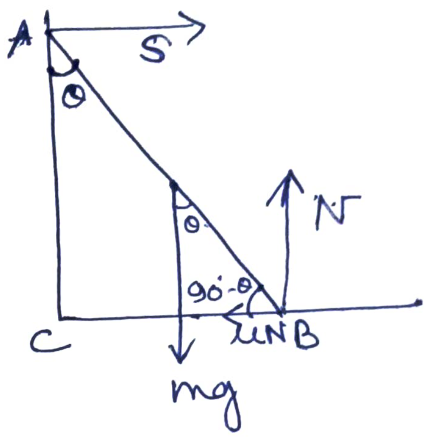
$$\sin 2C = \frac{c(a^2+b^2-c^2)}{2Rab}.$$

$$\therefore \frac{P}{\frac{a(b^2+c^2-a^2)}{2Rbc}} = \frac{Q}{\frac{b(c^2+a^2-b^2)}{2RCa}} = \frac{R}{\frac{c(a^2+b^2-c^2)}{2Rab}}$$

$$\therefore \frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}.$$

Q A uniform ladder rest in limiting equilibrium with lower end on a rough horizontal plane and upper end against a smooth vertical wall. If θ is the inclination of the wall to vertical then prove that

$$\tan\theta = 2\mu.$$



proof:- Let l be the length and mg be the weight of the ladder. AB.

~~At bottom A~~. Equations of motion is are.

vertically.

$$N = Mg.$$

Horizonally. $S = \mu N$

where N → Reaction and bottam point B.

S → " " upper " A.

Again taking moment about B.

$$Mg \cdot \frac{l}{2} \sin\theta - Sl\cos\theta = 0 \quad \text{[In equilibrium]}$$

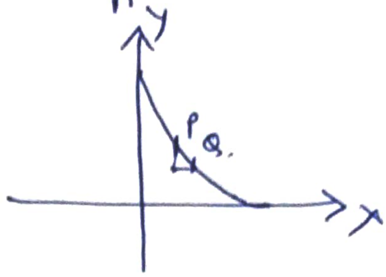
$$\Rightarrow \tan\theta = \frac{2S}{mg} = \cancel{2S}$$

$$= \frac{\mu N}{mg} = \frac{\mu mg}{mg}.$$

$$\Rightarrow \tan\theta = \mu.$$

#

Q Find the C.G of the arc of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ lying in first quadrant.



Soln:-

$$\bar{x} = \frac{\int_0^a x\,ds}{\int_0^a ds}$$

$ds \rightarrow$ length of the portion PQ

$$= \frac{\int_0^a x\sqrt{dx^2+dy^2}}{\int_0^a \sqrt{dx^2+dy^2}}$$

$$= \frac{\int_0^a x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\frac{\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}}$$

Here $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$

So $$\bar{x} = \frac{\int_0^a x\sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx}{\int_0^a \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx}$$

Let ~~$x = a\cos^3 t$~~

~~Then~~

Let $x = a\cos^3 t$.

Then $a^{2/3}\cos^2 t + y^{2/3} = a^{2/3}$

$\Rightarrow y^{2/3} = a^{2/3}\sin^2 t$

$\Rightarrow y = a\sin^3 t$

$dx = -a\,3\cos^2 t\sin t\,dt$

$$\therefore \bar{x} = \frac{\int_0^{\pi/2} a\cos^3 t\; a\,3\cos^2 t\sin t/(\cos t)\,dt}{\int_0^{\pi/2} 3\cos^2 t\sin t/\cos t\,dt}$$

$$= 2a/5.$$

Similarly $\bar{y} = \frac{2a}{5}$ (Calculate yourself).